Bago University
Department of Mathematics
First Semester Examination, March 2019

Fourth Year (B.Sc)
(Mathematics Specialization)

Math 4101
Analysis III
Time Allowed: (3) hours

Answer All Questions.

1. (a) Define the Riemann-Stieltjes Integral.
   (b) Prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exist a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

2. (a) Define a refinement of P. If P^* is a refinement of P, prove that $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.
   (b) If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on [a, b], prove that $fg \in \mathcal{R}(\alpha)$.

3. (a) Suppose $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, $\emptyset$ is continuous on $[m, M]$, and $h(x) = \emptyset(f(x))$ on $[a, b]$. Prove that $h \in \mathcal{R}(\alpha)$ on $[a, b]$.

   (b) Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and $\int f d\alpha = 0$.

4. (a) If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, prove that $|f|$ belong to $\mathcal{R}(\alpha)$ and
   $$\left|\int_a^b f \, d\alpha\right| \leq \int_a^b |f| \, d\alpha .$$

   (b) Suppose $f \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

5. (a) Define the unit step function. If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s and $\alpha(x) = I(x - s)$, prove that $\int_a^b f d\alpha = f(s)$.
   (b) If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

6. (a) Define total variation and bounded variation.
   (b) Suppose $\emptyset$ is strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\emptyset(y))$, $g(y) = f(\emptyset(y))$.
   Prove that $g \in \mathcal{R}(\beta)$ and $\int_A^B g d\beta = \int_a^b f d\alpha$.

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