Bago University

Department of Mathematics

Second Semester Examination, September 2019

Fourth Year (B.Sc) **Math-4107**

(Mathematics Specialization) **Analysis IV**

Answer All Questions. **Time Allowed: (3) hours**

1.(a) Define (i) pointwise convergence , (ii) uniform convergence , (iii) equicontinuous.

(b) Let $f_n(x) = n^2x(1-x^2)^n \quad (0 \le x \le 1,\ n = 1,2,3,\ldots)$.

Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx$.

2.(a) State and prove that the Stone-Weierstrass Test (or) M-Test Theorem.

(b) For $n = 1,2,3,\ldots$, define the function f_n by $f_n(x) = \left(\frac{1}{3}x\right)^n$ for $x \in [0,1]$. Show that the sequence $\{f_n\}$ converges uniformly on $[0,1]$ to the function f where $f(x) = 0$ for all $x \in [0,1]$.

3.(a) Let X be a metric space, E be a subset of X, f_n for $n = 1,2,3,\ldots$ and f be complex-valued functions defined on E and c be a complex number. If $\{f_n\}$ converges to f uniformly on E, prove that $\{c f_n\}$ uniformly converges to $c f$ on E.

(b) Let X be a metric space and $C(X)$ be the metric space of complex-value continuous bounded functions defined on X. If $f, g \in C(X)$, prove that $f+g, -g, f-g \in C(X)$.

4.(a) Suppose $\mathcal{A}$ is an algebra of functions on a set E, $\mathcal{A}$ separates points on E and $\mathcal{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E and c_1, c_2 are constants. Prove that $\mathcal{A}$ contains a function f such that $f(x_1) = c_1$ and $f(x_2) = c_2$.

P.T.O.

(b) Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

5.(a) If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E and $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n + g_n\}$ converges uniformly on E.

(b) Let α be a monotonically increasing function defined on a closed interval $[a,b]$, f_n for $n = 1,2,3,\ldots$ and f be complex-valued functions defined on $[a,b]$, $f_n \in \mathbb{R}(\alpha)$ on $[a,b]$, for each $n = 1,2,3,\ldots$ and $f_n \to f$ uniformly on $[a,b]$. Prove that $f \in \mathbb{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$.

6.(a) Let $\{f_n\}$ be a sequence of real-valued functions, differentiable on an interval $[a,b]$, for some $x_0 \in [a,b]$ the sequence $\{f_n(x_0)\}$ converges and let the sequence $\{f_n'\}$ converges uniformly on $[a,b]$. Show that $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$.

(b) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

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